Prime numbers and implication free reducts of MV$_n$-chains

Marcelo E. Coniglio$^1$, Francesc Esteva$^2$, Tommaso Flaminio$^2$, and Lluïs Godo$^2$

$^1$ Dept. of Philosophy - IFCH and CLE
University of Campinas, Campinas, Brazil
coniglio@cle.unicamp.br

$^2$ IIIA - CSIC, Bellaterra, Barcelona, Spain
{esteva,tommaso,godo}@iiia.csic.es

Abstract

Let L$_{n+1}$ be the MV-chain on the $n + 1$ elements set $L_{n+1} = \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\}$ in the algebraic language \{→, ¬\} [3]. As usual, further operations on L$_{n+1}$ are definable by the following stipulations: $1 = x \rightarrow x$, $0 = \neg 1$, $x \oplus y = \neg x \rightarrow y$, $x \odot y = \neg (\neg x \oplus \neg y)$, $x \land y = x \odot (x \rightarrow y)$, $x \lor y = \neg (\neg x \land \neg y)$. Moreover, we will pay special attention to the also definable unary operator $^*x = x \odot x$.

In fact, the aim of this paper is to continue the study initiated in [4] of the \{∗, ∨\}-reducts of the MV-chains L$_{n+1}$, denoted L$_n^\ast$. In fact L$_n^\ast$ is the algebra on L$_{n+1}$ obtained by replacing the implication operator → by the unary operation ∗ which represents the square operator $^*x = x \odot x$ and which has been recently used in [5] to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \langle L_4, \{1/3, 2/3, 1\}\rangle$. In this contribution we make a step further in studying the expressive power of the ∗ operation, in particular our main result provides a full characterization of those prime numbers $n$ for which the structures L$_{n+1}$ and L$_n^\ast$ are term-equivalent. In other words, we characterize for which $n$ the Lukasiewicz implication → is definable in L$_n^\ast$, or equivalently, for which $n$ L$_n^\ast$ is in fact an MV-algebra. We also recall that, in any case, the matrix logics (L$_n^\ast, F$), where $F$ is an order filter, are algebraizable.

Term-equivalence between L$_{n+1}$ and L$_n^\ast$

Let X be a subset of L$_{n+1}$. We denote by $\langle X\rangle^\ast$ the subalgebra of L$_n^\ast$ generated by X (in the reduced language \{∗, ∨\}). For $n \geq 1$ define recursively (∗)$^nx$ as follows: (∗)$^1x = ^*x$, and (∗)$^{i+1}x = ((^*)^ix)$, for $i \geq 1$.

A nice feature of the L$_n^\ast$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in L_{n+1} \mid a \leq b\}$, for every $a \in L_{n+1}$. A proof of the following result can be found in [4].

Proposition 1. For each $a \in L_{n+1}$, the unary operation $\Delta_a$ defined as

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in L$_n^\ast$. Therefore, for every $a \in L_{n+1}$, the operation $\chi_a$, i.e., the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

It is now almost immediate to check that the following implication-like operation is definable in every L$_n^\ast$: $x \Rightarrow y = 1$ if $x \leq y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0 \leq i \leq j \leq n} (\chi_i/n(x) \land \chi_j/n(y)).$$
Actually, one can also define Gödel implication on \( L_{n+1}^* \) by putting \( x \Rightarrow_G y = (x \Rightarrow y) \vee y \).

It readily follows from Proposition 1 that all the \( L_{n+1}^* \) algebras are simple as, if \( a > b \in L_{n+1} \) would be congruent, then \( \Delta_n(a) = 1 \) and \( \Delta_n(b) = 0 \) should be so. Recall that an algebra is called \textit{strictly simple} if it is simple and does not contain proper subalgebras. It is clear that if \( L_{n+1}^* \) and \( L_{n+1}^* \) are strictly simple, then \( \{0, 1\} \) is their only proper subalgebra.

\textbf{Remark 2.} It is well-known that \( L_{n+1}^* \) is strictly simple iff \( n \) is prime. Note that, for every \( n \), if \( B = (B, \neg, \rightarrow) \) is an MV-subalgebra of \( L_{n+1}^* \), then \( B^* = (B, \vee, \neg, *) \) is a subalgebra of \( L_{n+1}^* \) as well. Thus, if \( L_{n+1}^* \) is not strictly simple, then \( L_{n+1}^* \) is not strictly simple as well. Therefore, if \( n \) is not prime, \( L_{n+1}^* \) is not strictly simple. However, in contrast with the case of \( L_{n+1}^* \), \( n \) being prime is not a sufficient condition for \( L_{n+1}^* \) being strictly simple.

We now introduce the following procedure \( P \): given \( n \) and an element \( a \in L_{n+1}^* \setminus \{0, 1\} \), it iteratively computes a sequence \([a_1, \ldots, a_k, \ldots]\) where \( a_1 = a \) and for every \( k \geq 1 \),

\[ a_{k+1} = \begin{cases} \ast(a_k), & \text{if } a_k > 1/2 \\ \neg(a_k), & \text{otherwise (i.e., if } a_k < 1/2) \end{cases} \]

until it finds an element \( a_i \) such that \( a_i = a_j \) for some \( j < i \), and then it stops. Since everything is finite, the procedure always stops and produces a finite sequence. Then we write \( P(n, a) = \{a_1, a_2, \ldots, a_m\} \), where \( a_1 = a \) and \( a_m \) is such that \( P \) stops at \( a_m+1 \). Therefore,

\textbf{Lemma 3.} For each odd number \( n \), let \( a_1 = (n - 1)/n \). Then the procedure \( P \) stops after reaching \( 1/n \), that is, if \( P(n, a_1) = \{a_1, a_2, \ldots, a_m\} \) then \( a_m = 1/n \).

Furthermore, for any \( a \in L_{n+1}^* \setminus \{0, 1\} \), the set \( A_1 \) of elements reached by \( P(n, a) \), i.e. \( A_1 = \{b \in L_{n+1}^* \mid b \text{ appears in } P(n, a)\} \), together with the set \( A_2 \) of their negations, 0 and 1, define the domain of a subalgebra of \( L_{n+1}^* \).

\textbf{Lemma 4.} \( L_{n+1}^* \) is strictly simple iff \( \langle(n - 1)/n\rangle^* = L_{n+1}^* \).

\textit{Proof.} (Sketch) The ‘if’ direction is trivial. As for the other direction, call \( a_1 = (n - 1)/n \) and assume that \( \langle a_1 \rangle^* = L_{n+1}^* \). Launch the procedure \( P(n, a_1) \) and let \( A \) be the subalgebra of \( L_{n+1}^* \) whose universe is \( A_1 \cup A_2 \cup \{0, 1\} \) defined as above. Clearly \( a_1 \in A \), hence \( \langle a_1 \rangle^* \subseteq A \). But \( A \subseteq \langle a_1 \rangle^* \), by construction. Therefore \( A = \langle a_1 \rangle^* = L_{n+1}^* \).

\textbf{Fact:} Under the current hypothesis (namely, \( \langle a_1 \rangle^* = L_{n+1}^* \)) if \( n \) is even, then \( n = 2 \) or \( n = 4 \). Thus, assume \( n \) is odd, and hence Lemma 3 shows that \( 1/n \in A_1 \). Now, let \( c \in L_{n+1}^* \setminus \{0, 1\} \) such that \( c \neq a_1 \). If \( c \in A_1 \) then the process of generation of \( A \) from \( c \) will produce the same set \( A_1 \) and so \( A = L_{n+1}^* \), showing that \( \langle c \rangle^* = L_{n+1}^* \). Otherwise, if \( c \in A_2 \) then \( \neg c \in A_1 \) and, by the same argument as above, it follows that \( \langle c \rangle^* = L_{n+1}^* \). This shows that \( L_{n+1}^* \) is strictly simple.

\textbf{Lemma 5} ([4]). If \( L_{n+1}^* \) is term-equivalent to \( L_{n+1}^* \) then:

(i) \( L_{n+1}^* \) is strictly simple.

(ii) \( n \) is prime

\textbf{Theorem 6.} \( L_{n+1}^* \) is term-equivalent to \( L_{n+1}^* \) iff \( L_{n+1}^* \) is strictly simple.

\textit{Proof.} The ‘only if’ part is (i) of Lemma 5. For the ‘if’ part, since \( L_{n+1}^* \) is strictly simple then, for each \( a, b \in L_{n+1} \) where \( a \notin \{0, 1\} \) there is a definable term \( t_{a,b}(x) \) such that \( t_{a,b}(a) = b \). Otherwise, if for some \( a \notin \{0, 1\} \) and \( b \in L_{n+1} \) there is no such term then \( A = \langle a \rangle^* \) would be a
proper subalgebra of $L^*_{n+1}$ (since $b \not\in A$) different from \{0, 1\}, a contradiction. By Proposition 1
the operations $\chi_a(x)$ are definable for each $a \in L_{n+1}$, then in $L^*_{n+1}$ we can define Lukasiewicz
implication $\rightarrow$ as follows:

$$
x \rightarrow y = (x \Rightarrow y) \lor \left( \bigvee_{n>i>j\geq 0} \chi_{i/n}(x) \land \chi_{j/n}(y) \land t_{i/n.a_{ij}}(x) \right) \lor \left( \bigvee_{n>j\geq 0} \chi_1(x) \land \chi_{j/n}(y) \land y \right)
$$

where $a_{ij} = 1 - i/n + j/n$.

We have seen that $n$ being prime is a necessary condition for $L_{n+1}$ and $L^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers $n$ for which $L_{n+1}$ and $L^*_{n+1}$ are not term-equivalent and this is the case, for instance, of $n = 17$.

**Definition 7.** Let $\Pi$ be the set of odd primes $n$ such that $2^m$ is not congruent with $\pm 1 \bmod n$ for all $m$ such that $0 < m < (n-1)/2$.

Since, for every odd prime $n$, $2^m$ is congruent with $\pm 1 \bmod n$ for $m = (n-1)/2$ then $n$ is in $\Pi$ iff $n$ is an odd prime such that $(n-1)/2$ is the least $0 < m$ such that $2^m$ is congruent with $\pm 1 \bmod n$.

The following is our main result and it characterizes the class of prime numbers for which the Lukasiewicz implication is definable in $L^*_{n+1}$.

**Theorem 8.** For every prime number $n > 5$, $n \in \Pi$ iff $L_{n+1}$ and $L^*_{n+1}$ are term-equivalent.

The proof of theorem above makes use of the procedure $P$ defined above. Let $a_1 = (n-1)/n$ and let $P(n, a_1) = [a_1, \ldots, a_i]$. By the definition of the procedure $P$, the sequence $[a_1, \ldots, a_i]$ is the concatenation of a number $r$ of subsequences $[a_{i_1}, \ldots, a_{i_{r_1}}], [a_{i_2}, \ldots, a_{i_{r_2}}], \ldots, [a_{i_r}, \ldots, a_{i_{r_r}}]$, with $a_{i_1} = a_l$ and $a_{i_r} = a_l$, where for each subsequence $1 \leq j \leq r$, only the last element $a_{i_j}$ is below 1/2, while the rest of elements are above 1/2.

Now, by the very definition of $\ast$, it follows that the last elements $a_{i_{j_i}}$ of every subsequence are of the form

$$
a_{i_{j_i}} = \begin{cases} 
k n - 2^m \bmod n, & \text{if } j \text{ is odd} \\
2^m - kn \bmod n, & \text{otherwise, i.e. if } j \text{ is even}
\end{cases}
$$

for some $m, k > 0$, where in particular $m$ is the number of strictly positive elements of $L_{n+1}$ which are obtained by the procedure before getting $a_{i_{j_i}}$.

Now, Lemma 3 shows that if $n$ is odd then 1/n is reached by $P$, i.e. $a_l = a_{i_{r_l}} = 1/n$. Thus,

$$
\begin{cases} 
k n - 2^m = 1, & \text{if } r \text{ is odd (i.e., } 2^m \equiv -1 \bmod n) \text{ if } r \text{ is odd} \\
2^m - kn = 1, & \text{otherwise (i.e., } 2^m \equiv 1 \bmod n) \text{ if } r \text{ is even}
\end{cases}
$$

where $m$ is now the number of strictly positive elements in the list $P(n, a_1)$, i.e. that are reached by the procedure.

Therefore $2^m$ is congruent with $\pm 1 \bmod n$. If $n$ is a prime such that $L^*_{n+1}$ is strictly simple, the integer $m$ must be exactly $(n - 1)/2$, for otherwise $\langle a_1 \rangle^*$ would be a proper subalgebra of $L^*_{n+1}$ which is absurd. Moreover, for no $m' < m$ one has that $2^{m'}$ is congruent with $\pm 1 \bmod n$ because, in this case, the algorithm would stop producing a proper subalgebra of $L^*_{n+1}$. This result, together with Theorem 6, shows the right-to-left direction of Theorem 8.
In order to show the other direction assume, by Theorem 6, that $L_{i,n+1}^*$ is not strictly simple. Thus, by Lemma 4, $(a_1)^*_{i,n}$ is a proper subalgebra of $L_{i,n+1}^*$ and hence the algorithm above stops, in $1/n$, after reaching $m < (n-1)/2$ strictly positive elements of $L_{i,n+1}^*$. Thus, $2^m$ is congruent with $\pm 1$ (depending on whether $r$ is even or odd, where $r$ is the number of subsequences in the list $P(n,a_1)$ as described above) mod $n$, showing that $n \not\in \Pi$.

**Algebraizability of** $\langle L_{i,n+1}^*, F_{i,n} \rangle$

Given the algebra $L_{n+1}^*$, it is possible to consider, for every $1 \leq i \leq n$, the matrix logic $L_{i,n+1}^* = \langle L_{i,n+1}^*, F_{i,n} \rangle$. In this section we recall from [4] that all the $L_{i,n+1}^*$ logics are algebraizable in the sense of Blok-Pigozzi [1], and that, for every $i, j$, the quasivarieties associated to $L_{i,n+1}^*$ and $L_{j,n+1}^*$ are the same.

Observe that the operation $x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise is definable in $L_{n+1}^*$. Indeed, it can be defined as $x \approx y = (x \Rightarrow y) \land (y \Rightarrow x)$. Also observe that $x \approx y = \Delta_1((x \Rightarrow_G y) \land (y \Rightarrow_G x))$ as well.

**Lemma 9.** For every $n$, the logic $L_{n+1}^* := L_{n,n+1}^* = \langle L_{n+1}^*, \{1\} \rangle$ is algebraizable.

**Proof.** It is immediate to see that the set of formulas $\Delta(p,q) = \{p \Rightarrow q\}$ and the set of pairs of formulas $E(p,q) = \{\langle p, \Delta_0(p) \rangle\}$ satisfy the requirements of algebraizability. \hfill \Box

Blok and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems $S_1$ and $S_2$ in the same language are equivalent if there are translations $\tau_i : S_i \rightarrow S_j$ for $i \neq j$ such that: $\Gamma \vdash_{S_i} \varphi \iff \Gamma \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv \vdash_{S_i} \tau_i(\varphi))$. From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the algebraic point of view, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This can be applied to $L_{i,n+1}^*$.

**Lemma 10.** For every $n$ and every $1 \leq i \leq n - 1$, the logics $L_{i,n+1}^*$ and $L_{i,n+1}^*$ are equivalent.

Indeed, it is enough to consider the translation mappings $\tau_1 : L_{i,n+1}^* \rightarrow L_{i,n+1}^*$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : L_{i,n+1}^* \rightarrow L_{i,n+1}^*$, $\tau_{i,2}(\varphi) = \Delta_i(\varphi)$. Therefore, as a direct consequence of Lemma 9, Lemma 10 and the observations above, it follows the algebraizability of $L_{i,n+1}^*$.

**Theorem 11.** For every $n$ and for every $1 \leq i \leq n$, the logic $L_{i,n+1}^*$ is algebraizable.

Therefore, for each logic $L_{i,n+1}^*$ there is a quasivariety $Q(i,n)$ which is its equivalent algebraic semantics. Moreover, by Lemma 10 and by Blok and Pigozzi’s results, $Q(i,n)$ and $Q(j,n)$ coincide, for every $i, j$. The question of axiomatizing $Q(i,n)$ is left for future work.

**References**


