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On an implication-free reduct of $MV_n$ chains

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Abstract

Let $L_{n+1}$ be the MV-chain on the $n + 1$ elements set $L_{n+1} = \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\}$ in the algebraic language $\{\to, \neg\}$ [2]. As usual, further operations on $L_{n+1}$ are definable by the following stipulations: $1 = x \to x$, $0 = \neg 1$, $x \oplus y = \neg x \to y$, $x \odot y = \neg (\neg x \oplus \neg y)$, $x \land y = x \odot (x \to y)$, $x \lor y = \neg (\neg x \land \neg y)$. Moreover, we will pay special attention to the also definable unary operator $*x = x \odot x$.

In fact, the aim of this paper is to study the $\{*, \neg, \lor\}$-reducts of the MV-chains $L_{n+1}$, that will be denoted as $L_{n+1}^*$, i.e. the algebra on $L_{n+1}$ obtained by replacing the implication operator $\to$ by the unary operation $*$ which represents the square operator $*x = x \odot x$ and which has been recently used in $[1]$ to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \{\mathcal{L}_4, \{1/3, 2/3, 1\}\}$. In this contribution we make a step further in studying the expressive power of the $*$ operation, in particular we will focus on the question for which natural numbers $n$ the structures $L_{n+1}$ and $L_{n+1}^*$ are term-equivalent. In other words, for which $n$ the Lukasiewicz implication $\to$ is definable in $L_{n+1}^*$, or equivalently, for which $n$ $L_{n+1}^*$ is in fact an MV-algebra. We also show that, in any case, the matrix logics $(L_{n+1}^*, F)$, where $F$ is an order filter, are algebraizable. What we present here is a work in progress.

Term-equivalence between $L_{n+1}$ and $L_{n+1}^*$

Let $X$ be a subset of $L_{n+1}$. We denote by $(X)^*$ the subalgebra of $L_{n+1}^*$ generated by $X$ (in the reduced language $\{*, \neg, \lor\}$). For $n \geq 1$ define recursively $(*)^n x$ as follows: $(*)^0 x = x$ and $(*)^{i+1} x = (*)(*)^i x$, for $i \geq 1$.

A nice feature of the $L_{n+1}^*$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in L_{n+1} \mid a \leq b\}$, for every $a \in L_{n+1}$.

Proposition 1. For each $a \in L_{n+1}$, the unary operation $\Delta_a$ defined as

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in $L_{n+1}^*$. As a consequence, for every $a \in L_{n+1}$, the operation $\chi_a$ that corresponds to the characteristic function of $a$ (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

Proof. The case $a = 1$ corresponds to the Monteiro-Bazg Delta operator and, as is well-known, it can be defined as $\Delta_1(x) = (*)^0 x$. For $a = 0$ define $\Delta_a(x) = \Delta_1(x) \lor \Delta_1(x)$; then $\Delta_a(x) = 1$ for every $x$. Now, assume $0 < a = i/n < 1$. It is not difficult to show that one can always find a sequence of terms (operations) $t_1(x), \ldots, t_m(x)$ over $\{*, \neg\}$ such that $t_1(t_2(\ldots(t_m(x)\ldots))) = 1$.
Lemma 4. As for the operations $\chi_a$, define $\chi_1 = \Delta_1$, $\chi_0 = -\Delta_1/n$, and if $0 < a < 1$, then define $\chi_a = \Delta_a \land -\Delta_{a-1/n}$.

It is now almost immediate to check that the following implication-like operation is definable in every $L^*_{n+1}$: $x \Rightarrow y = 1$ if $x \leq y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0 \leq i \leq j \leq n} (\chi_{i/n}(x) \land \chi_{j/n}(y)).$$

Actually, one can also define Gödel implication on $L^*_{n+1}$ by putting $x \Rightarrow y = (x \Rightarrow y) \lor y$.

On the other hand, it readily follows from Proposition 1 that all the $L^*_{n+1}$ algebras are simple. Indeed, if $a > b \in L^*_{n+1}$ would be congruent, then $\Delta_n(a) = 1$ and $\Delta_n(b) = 0$ should be so. Recall that an algebra is called strictly simple if it is simple and does not contain proper subalgebras. It is clear then that the case of $L^*_{n+1}$ and $L^*_{n+1}$ algebras, they are strictly simple if $(0, 1)$ is their only proper subalgebra.

Remark 2. It is well-known that $L^*_{n+1}$ is strictly simple iff $n$ is prime. Note that, for every $n$, if $B = (B, \lor, \land, \neg)$ is an MV-subalgebra of $L^*_{n+1}$, then $B^* = (B, \lor, \land, \neg)$ is a subalgebra of $L^*_{n+1}$ as well. Thus, if $L^*_{n+1}$ is not strictly simple, then $L^*_{n+1}$ is not strictly simple as well. Therefore, if $n$ is not prime, $L^*_{n+1}$ is not strictly simple. However, in contrast with the case of $L^*_{n+1}$, $n$ being prime is not a sufficient condition for $L^*_{n+1}$ being strictly simple. In Lemma 7 below we will provide some examples of prime $n$ for which $L^*_{n+1}$ is not strictly simple, in view of Theorem 6.

**Lemma 3.** $L^*_{n+1}$ is strictly simple iff $\langle (n-1)/n \rangle^* = L^*_{n+1}$.

**Proof.** The 'only if' direction is trivial. In order to prove the converse, assume that $\langle a_1 \rangle^* = L^*_{n+1}$ for $a_1 = (n-1)/n$. For $i \geq 1$ let $a_{i+1} = t_i(a_1)$ such that $t_i(x) = *x$ if $x > 1/2$, and $t_i(x) = -*x$ otherwise. Since $L^*_{n+1}$ is finite, there is $1 \leq i < j$ such that $a_i = a_j$ and so $A_1 := \{a_1 \mid i \geq 1\} = \{a_1 \mid 1 \leq i \leq k\}$ for some $k$ such that $a_1 \neq a_j$ if $1 \leq i, j \leq k$. Let $A = A_1 \cup A_2 \cup \{0, 1\}$ where $A_2 = \{\neg a \mid a \in A_1\}$. Since $*1 = 1$ and $*x = 0$ if $x \leq 1/2$, $A$ is the domain of a subalgebra $A$ of $L^*_{n+1}$ over $\langle *, \land, \lor \rangle$ such that $a_1 \in A$, hence $\langle a_1 \rangle^* \subseteq A$. But $A \subseteq \langle a_1 \rangle^*$, by construction. Therefore $A = \langle a_1 \rangle^* = L^*_{n+1}$.

**Fact:** Under the current hypothesis (namely, $\langle a_1 \rangle^* = L^*_{n+1}$): if $n$ is even then $n = 2$ or $n = 4$. Indeed, suppose that $\langle a_1 \rangle^* = L^*_{n+1}$ and $n$ is even. If $n = 2$ or $n = 4$ then clearly $L^*_{n+1}$ is strictly simple. Now, assume $n > 4$. Observe that: (1) for any $a \in L^*_{n+1} \setminus \{0, 1\}$, $*a = i/n$ such that $i$ is even; and (2) if $i < n$ is even then $(-i/n) = (n - i)/n$ such that $n - i$ is even. That being so, if $i/n \in (A_1 \cup A_2) \setminus \{a_1, -a_1\}$ (recall the process described above) then $i$ is even. But then, for instance, $3/n \notin A = \langle a_1 \rangle^* = L^*_{n+1}$, a contradiction. This proves the Fact.

From the Fact, assume now that $n$ is odd, and let $a = ((n + 1)/2)/n$ and $b = ((n - 1)/2)/n$. Since $-a = b$, $-b = a$ and $a, b \in A$ then, by construction of $A$, there is $1 \leq k$ such that either $a = a_k$ or $b = a_k$. If $a = a_k$ then $a_{k+1} = *a = 1/n$ and so $a_{k+2} = -a_{k+1} = -1/n = (n-1)/n = a_1$. Analogously it can be proven that, if $b = a_k$ then $a_{j+1} = a_j$ for some $j > i$. This shows that $A_1 = \{a_1, \ldots, a_k\}$ is such that $a_{k+1} = a_1$ (hence $a_k = 1/n$). Now, let $c \in L^*_{n+1} \setminus \{0, 1\}$ such that $c \neq a_1$. If $c \in A_1$ then the process of generation of $A$ from $c$ will produce the same set $A_1$ and so $A = L^*_{n+1}$, showing that $\langle c \rangle = L^*_{n+1}$. Otherwise, if $c \in A_2$ then $\neg c \in A_1$ and, by the same argument as above, it follows that $\langle c \rangle = L^*_{n+1}$. This shows that $L^*_{n+1}$ is strictly simple.

**Lemma 4.** If $L^*_{n+1}$ is term-equivalent to $L^*_{n+1}$ then $L^*_{n+1}$ is strictly simple.
Proof. If $L_{n+1}$ is term-equivalent to $L^*_{n+1}$ then $\odot$ is definable in $L^*_{n+1}$, and hence $\langle (n-1)/n \rangle = L^*_{n+1}$. Indeed, we can obtain $(n-i-1)/n = ((n-1)/n) \odot ((n-i)/n)$ for $i = 1, \ldots, n-1$, and $1 = -0$. By Lemma 3 it follows that $L^*_{n+1}$ is strictly simple.

Corollary 5. If $L_{n+1}$ is term-equivalent to $L^*_{n+1}$ then $n$ is prime.

Proof. If $L_{n+1}$ is term-equivalent to $L^*_{n+1}$ then $L^*_{n+1}$ is strictly simple, by Lemma 4. By Remark 2 it follows that $n$ must be prime.

Theorem 6. $L_{n+1}$ is term-equivalent to $L^*_{n+1}$ iff $L^*_{n+1}$ is strictly simple.

Proof. The ‘only if’ part is Lemma 4. For the ‘if’ part, since $L^*_{n+1}$ is strictly simple then, for each $a, b \in L_{n+1}$ where $a \notin \{0, 1\}$ there is a definable term $t_{a,b}(x)$ such that $t_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0, 1\}$ and $b \in L_{n+1}$ there is no such term then $A = \{a\}^*$ would be a proper subalgebra of $L^*_{n+1}$ (since $b \notin A$) different from $\{0, 1\}$, a contradiction. By Proposition 1 the operations $\chi_a(x)$ are definable for each $a \in L_{n+1}$; then in $L^*_{n+1}$ we can define Lukasiewicz implication $\rightarrow$ as follows:

$$x \rightarrow y = (x \Rightarrow y) \lor \left( \bigvee_{n>j>0} \chi_{i/n}(x) \land \chi_{j/n}(y) \land t_{i/n,a,j}(x) \right) \lor \left( \bigvee_{n>j>0} \chi_{1}(x) \land \chi_{j/n}(y) \land y \right)$$

where $a_{ij} = 1 - i/n + j/n$.

We have seen that $n$ being prime is a necessary condition for $L_{n+1}$ and $L^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers $n$ for which $L_{n+1}$ and $L^*_{n+1}$ are not term-equivalent.

Lemma 7. If $n$ is a prime Fermat number greater than 5 then $L_{n+1}$ and $L^*_{n+1}$ are not term-equivalent.

Proof. Recall that a Fermat number is of the form $2^{2^k} + 1$, with $k$ being a natural number. We are going to prove that if $n$ is a prime Fermat number and $a_1 = (n-1)/n$, then $\langle a_1 \rangle^*$ is a proper subalgebra of $L^*_{n+1}$ (recall Theorem 6 and Lemma 3). Thus, let $n > 5$ be a prime Fermat number, that is, a prime number of the form $n = 2^m + 1$ with $m = 2^k$ and $k > 1$. The $(m-1)$-times iterations of $\ast$ applied to $a_1$ produce $((n+1)/2)/n$, that is: $\langle (\ast)^{m-1}(a_1) = ((n+1)/2)/n$. Since $\ast(((n+1)/2)/n) = 1/n$, the constructive procedure for generating the algebra $\langle a_1 \rangle^*$ described in the proof of Lemma 3 shows that $\langle a_1 \rangle^* = A$ has $2m + 2$ elements: $m$ elements in $A_1$, plus $m$ elements in $A_2$ corresponding to their negations, plus 0 and 1. Since $2m+2 < 2^m+1 = n$ as $n > 5$, $\langle a_1 \rangle^*$ is properly contained in $L_{n+1}$, and it is different from $\{0, 1\}$.

The first Fermat prime number greater than 5 is $n = 17$. It is easy to see that $\langle 16/17 \rangle^* = \{0, 1/17, 2/17, 4/17, 8/17, 9/17, 13/17, 15/17, 16/17, 1\}$.

Actually, we do not have a full characterisation of those prime numbers $n$ for which $L_{n+1}$ and $L^*_{n+1}$ are term-equivalent. But computational results show that for prime numbers until 8000, about 60% of the cases yield term-equivalence.
Algebraizability of \( \langle L_{n+1}^*, F_i/n \rangle \)

Given the algebra \( L_{n+1}^* \), it is possible to consider, for every \( 1 \leq i \leq n \), the matrix logic \( L_{i,n+1}^* = \langle L_{i,n+1}^*, F_i/n \rangle \). In this section we will shown that all the \( L_{i,n+1}^* \) are algebraizables in the sense of Blom-Pigozzi [1], and the quasivarieties associated to \( L_{i,n+1}^* \) and \( L_{i,n+1,1}^* \) are the same, for every \( i,j \).

Observe that the operation \( x \approx y = 1 \) if \( x = y \) and \( x \approx y = 0 \) otherwise is definable in \( L_{n+1}^* \). Indeed, it can be defined as \( x \approx y = (x \Rightarrow y) \land (y \Rightarrow x) \). Also observe that \( x \approx y = \Delta_1((x \Rightarrow_G y) \land (y \Rightarrow_G x)) \) as well.

In order to prove the main result of this section, we state the following:

**Lemma 8.** For every \( n \), the logic \( L_{n+1}^*_n := L_{i,n+1}^* = \langle L_{n+1}^*, \{1\} \rangle \) is algebraizable.

**Proof.** It is immediate to see that the set of formulas \( \Delta(p,q) = \{p \approx q\} \) and the set of pairs of formulas \( E(p,q) = \{(p, \Delta_0(p))\} \) satisfy the requirements of algebraizability.

Blom and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems \( S_1 \) and \( S_2 \) in the same language are equivalent if there are translations \( \tau_i : S_i \rightarrow S_j \) for \( i \neq j \) such that: \( \Gamma \vdash_{S_i} \varphi \iff \tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi) \), and \( \varphi \vdash_{S_i} \tau_i(\varphi) \).

From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the point of view of algebra, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This will be applied to \( L_{i,n+1}^* \).

**Lemma 9.** The logics \( L_{i,n+1}^* \) and \( L_{i,n+1}^* \) are equivalent, for every \( n \) and for every \( 1 \leq i \leq n-1 \).

**Proof.** It is enough to consider the translation mappings \( \tau_1 : L_{i,n+1}^* \rightarrow L_{i,n+1}^*, \tau_1(\varphi) = \Delta_1(\varphi) \), and \( \tau_2 : L_{i,n+1}^* \rightarrow L_{i,n+1}^*, \tau_2(\varphi) = \Delta_{i/n}(\varphi) \).

Finally, as a direct consequence of Lemma 8, Lemma 9 and the observations above, we can prove the following result.

**Theorem 10.** For every \( n \) and for every \( 1 \leq i \leq n \), the logic \( L_{i,n+1}^* \) is algebraizable.

As an immediate consequence of Theorem 10, for each logic \( L_{i,n+1}^* \) there is a quasivariety \( Q(i,n) \) which is its equivalent algebraic semantics. Moreover, by Lemma 9 and by Blom and Pigozzi’s results, \( Q(i,n) \) and \( Q(j,n) \) coincide, for every \( i,j \). The question of axiomatising \( Q(i,n) \) is left for future work.

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